ordered and unordered quantifiers

[DRAFT – 12 April 2015]

introduction: quantifiers

Many, perhaps most, quantified sentences, contain more than one quantifier, especially if we take account of implicit quantification in constructions that are not expressed as determiners, such as tenses and adverbs of quantification. ("Usually, when a cat has chased most of the mice, it then gets bored and looks for a bird or two" - five quantifiers.) And iterated quantifiers are at the heart of our abilities to express complex thoughts. Yet the order of quantifiers is not apparent from the surface syntax of many sentences, and it is not easy to extract it from more sophisticated considerations [FTN]. The purpose of this paper is to describe a way to think of quantifiers as fundamentally unordered, in a way that allows the information usually given by order, by specifying which quantifier is in the scope of which, to be extracted from further considerations. This can result in a traditional linear quantifier ordering, or as the more general structures of branching quantifiers, or as quantifiers with no ordering at all. One consequence of this approach is that it reveals more ordering possibilities with 'generalized quantifiers', quantifiers besides those definable in terms of the standard universal or existential quantifiers.

In the remainder of this section I sketch the basic aspects of the approach, leaving details and applications to later sections.

Restrictions on quantifiers are central to the account I shall present. A restriction states what smaller part of the whole domain of discourse figures in a quantifier's contribution to a sentence's truth value. The restrictions of universal and existential quantifiers are often absorbed into their main content. Thus "all cats hate getting wet" is equivalent to "if anything is a cat it hates getting wet". But philosophical favorites though they may be, the universal and existential quantifiers are exceptions in this regard. "Most cats hate getting wet" is not equivalent to "if anything is a cat then it hates getting wet. (Suppose there are seven individuals in the domain, of which three are cats but only one hates getting wet. Then "most cats hate getting wet" is false, but "most individuals are such that if they are cats then they hate getting wet" is true, at least when the conditional is material, since most individuals are not cats.) In [an] appendix I give counterexamples to a series of facts about ALL that do not hold for MOST. (And fail for many other quantifiers: MOST is a convenient foil for ALL in part because of the simplicity of the counterexamples.) And in later sections of the paper I make a number of remarks to lessen our confidence that restrictions can be absorbed into the main content even for ALL and SOME.

[FTN: A first reaction might be that the problem is an artifact of taking the conditional to be material. But this is what the material conditional is best for: if you were designing a connective to express a restriction to a smaller domain it is what you would come up with. All the same, as a matter of English idiom we do say things such as "mostly, if it's a cat it doesn't like the rain, or "if it's a typical cat, it chases mice". I suspect these are adaptations of expressions that are literally true for other quantifiers, such as *all*]

The general idea for deriving ordered quantifiers from an unordered basis can be given with a standard example. Given a two place relation, such as "chases" we usually specify how many chase and how many each chaser chases by inserting "some" and "all", in the familiar way derided by logic teachers, in the argument places of the relation, as in "everyone chases someone". This sentence is notoriously ambiguous: it specifies that every chaser has a chasee, but not how many chasees there are. If we do not specify this then we know only that there is a chasee for each chaser. ("Everyone chases", we say, meaning everyone chases someone or other, and this default reading makes sense since you cannot be a chaser without chasing something.) But we can also specify that one chasee is all that is needed to make the assertion true. Then we know that there is an individual chased by everyone. So the difference is between "each individual chases one or more individuals" and "each individual chases an individual that is the same one in all cases". The first of these is of the "for all x there is a y" form, and the second of the reversed "there is a y such that for all x".

When the origin of quantifier ordering is seen this way, the standard universal and existential quantifiers look even more exceptional. For we can just as easily specify that the chasee role is divided between seven or a hundred dogs, as in "each cat chases one of seven dogs"("there are seven dogs, and each cat chases one of them".) We give such qualifications when we need to be very clear about what we mean, as when we want to reverse the most likely interpretations of the syntactically parallel "someone ate all the sandwiches" and "someone dies every minute". We can underline the less likely interpretation of the first by saying "some person or persons ate all the sandwiches", where the possible plurality makes it clear that no single eater is being postulated. We can underline the extremely unlikely interpretation of the second by saying "some single person dies every minute".

Ftn: refs, and suggestion that when people hear the three Lincoln sentences they register the contrast, but not which of the first two is which.

The device can be applied to a range of quantifiers. For example in "most (of the) cats chase most (of the) dogs", taken so that neither quantifier has scope over the other – so the information is just "there is chasing: most of the cats are in the chasing role and most of the dogs are in the chasee role" – we can give "most cats" wider scope by requiring in addition that there be a set C with a majority of the cats in it and a set D with a majority of dogs in it, such that for every member of C the set of dogs that it chases be D, and that there be only n>0 such majority sets of cats. If n=1 there are examples in a three-element domain, where "Most.x Most.y Rxy" is true under this interpretation but "Most.y Most.x Rxy" is false, and moreover "Most.x Most.y Rxy" understood as "Most x are such that they have R to a majority of y" is false. (R= {(1,1), (1,2),(2,1), (2,3) .) With other quantifiers similar facts hold. For example there are examples in a four-element domain where "2x.3y Rxy" is false with n=1, although "for 2 individuals x it is the case that each has R to 3 individuals" is true. (R = {(1,1), (1,2), (1,3), (2,1), (2,2), (2,4)} .)

These examples illustrate two basic points. First, there are two distinct though related functions of quantifiers. One is to specify how many individuals satisfy a

monadic open sentence or an argument place of a relation. A standard position for such a quantifier in logical notation is as the first of a string of linearly ordered quantifiers, where only one argument place remains unfilled. Call this the absolute function of quantifiers. It contrasts with the relative function, in which a quantifier specifies how many individuals are related in a specified way *to* an individual, typically one satisfying a condition stated in terms of another quantifier. This pattern of relative quantifiers within the scopes of absolute quantifiers arises naturally from the Frege-Tarski tradition, on which quantifiers are successively applied to open sentences of decreasing adicity. It is this decreasing adicity feature which is responsible for many of the counterintuitive properties of quantifiers such as "most" when conceived in this way, as detailed in [the] appendix.
[FTN: the universal quantifier is unusual in that it is the same in both absolute and relative form: when members of a subset of the domain have a relation to everything, then the number of individuals occupying that place of the relation is all of the domain.]

The second basic point is that there is information about the structure of situations or models making a sentence true that is not given either by quantifiers in their absolute or their relative function alone. For example "most cats chase most dogs", taken as requiring that there be one majority set of cats all of whom are chasers, is not equivalent to "most cats are such that they chase most dogs". In this connection note also that with a sentence such as "six cats chase five dogs" we do not get the Frege-Tarski interpretation "six cats are such that each of them chases five dogs" just by imposing a cardinality condition on the wider scope quantifier. Requiring that there are six cats that chase does not force it to be equivalent to "six cats are such that they each chase five dogs", since this latter can also be true when there are five dogs, each of which is chased by one of six cats.

The Frege-Tarski tradition can be contrasted with the Mostowski-Lindstrom tradition, in which quantifiers are taken directly as properties of sets of n-tuples, typically those which are closed under isomorphism of models, which may or may not be describable in terms of an iteration of operations on single argument places.

An example of a quantifier that is easily conceived of in M-L terms but hard to understand in F-T terms is Qxy defined by "Qxy Rxy is true when there is no 1-1 correspondence between the pairs (x,y) for which Rxy and the pairs (x,y) for which Ryx". Quantifiers are easier to define, and definitions of truth and satisfaction are easier to give, on the F-T approach, but the M-L approach gives a greater range of possible quantifiers. The two approaches are related to the two functions of quantifiers in that the F-T approach makes the relative function inevitable while on the M-L approach there are quantifiers which have no relative aspect, for example the quantifier Qxy defined by "Qxy Rxy is true when there are infinitely many pairs (x,y) for which Rxy." So while the F-T "such that" interpretation is a perfectly intelligible way to understand some quantifier combinations it is not the only way. Independent specification of how many are required in each argument place, and how many of the individuals filling one place of a relation have the relation to how many filling another, will often be needed.

We are so accustomed to the F-T approach that we tend to ignore the fact that it fails to state, or leaves to context, information that can be crucial to thought and communication. "Some cats chase some dogs" does not specify how many are chasing or being chased or how many ways the dog that a given cat chases may be chosen. And if we do specify these facts for, for example, "Six cats are chasing five dogs", then a standard meaning, and possibly the default one, is that there are six chasing cats and five chased dogs, while each of the six chases some contextually specified number of the five and each of the five is chased by some contextually specified number of the six. This can happen with "all" and "some" too: sometimes when we say, for example, "all the cats are chasing all the dogs" we mean that all the cats are chasing and all the dogs being chased, leaving it open or to be settled by context how many dogs each cat chases and how many cats each dog is chased by. Neither "all" is then in the scope of the other, and the quantifiers are unordered. So one possibility for unordered quantifiers is that they simply do not specify how many individuals each individual satisfying their argument places (the quantifiers 'witnesses') has the relation to, any more than ordered quantifiers specify how many individuals satisfy the argument place bound by the inner-scope quantifiers. Some information is left out either way. In the rest of this paper I will develop these ideas in more detail.

[FTN: the 'double absolute' reading of a two quantifier sentence is encouraged in English by using progressive tenses ('are/were/will be chasing'). The reason is that these tenses suggest that the events took place over an interval, during which there is time for e.g. each of several cats to chase each of several dogs successively. So too with references to longer spans of time ("last month".) One would expect similar consequences with e.g imperfect tenses in Romance languages.]

FTN? where? When Q_1 and Q_2 are "none" or "exactly one" then the range of interpretations/readings is [drastically] reduced, to the two ordered interpretations "Q_1x are such that for Q_2y Rxy" and "Q_2y are such that for Q_1x Rxy". The reason for "none" is that if no As have R to even one B then none have R to any greater number. The reason for "exactly one" is that if *a* is the A that has R to some Bs, then there is a determinate number of B that *a* has R to, so that greater numbers suggest false claims. ("Exactly one cat is chasing dogs" does bear a variety of interpretations, though. Perhaps that cat chases all the dogs, perhaps even just one of them, most likely a few of them.) This fact is significant in that it shows one way in which an ordering can be required by the nature of the quantifiers, beginning from a form that is not intrinsically ordered. It should make us want arguments rather than assuming that ordered and unordered quantifier prefixes must have [fundamentally different origins.] This is relevant to a widely accepted claim of Barwise's discussed in section ///.

restrictions

In the next section I apply facts about restrictions to the ordering of quantifiers. In stating these facts, I use standard ideas about scope, usually applied to a sentence built around a binary relation. I shall write

$\{Q_1x, Q_2y\}$ Rxy
Ax, Ay

for the application of quantifiers Q_1 and Q_2y to R, where Q_1 is restricted to A and Q_2 to B. Ax and By are the restrictions and Rxy is the scope clause. I use the brackets {} to indicate that the quantifiers are unordered, so there is no significance to the order in which they are written. I shall often name quantifiers with capital letters – ALL, SOME, MOST etc. – to block the associations tied to standard logical symbols.

FTN Type (1,1) rather than type (1), in the terminology of Peters and Westerstahl. (The terminology comes from Lindstrom.) There are also more complex types.

The restriction for "Most" cannot be absorbed into the scope clause as a conditional or a conjunction, as already mentioned. So it is in relational expressions too: "Most cats chase all dogs" needs the proportion of the cats that chase all the dogs to be compared to the number of cats, not the number of cats-and-dogs or the domain generally. (The same is true of e.g. "Most cats chase a few of the dogs", but these sentences need to be handled wth care.) The same is true of just about all quantifiers which express the proportion of individuals that have some attribute. *At least a third, nearly all, comparatively few, many*, and others. (Some, such as *[a] few* or *too many* indicate proportionality in some contexts and not in others.) *All* is again on the standard treatment an exception, since it states a proportion, 100%, while allowing "all A are B" to be expressed in terms of a condition on the larger domain alone. But in linguistics, ALL is taken to need a restriction too. If ALL is a typical quantifier and occurs in the syntactical contexts where other quantifiers are found, which the standard semantics will refer to, is dyadic, then it is best treated as two-place (restriction plus scope clause) also. There is also a semantical argument for taking *all* as dyadic, of a kind more familiar to philosophers. We want to have truth conditions for sentences that combine ALL with quantifiers such as MOST, and the combination can force ALL to have a restriction separate from its scope clause. The clearest examples are where the quantifiers share a restriction. Consider for example "Most lovers, whoever it is they love, later quarrel with them", taken so that "loves" is the common restriction to the MOST and ALL ("whoever") quantifiers. The form is then

MOST x ALL y Quarrel xy
Love xy

Taking this in F-T style, it is true when taking most first members of a pair such that x loves y it is the case that whatever the second member of the pair, the first and second members later quarrel. This is equivalent to neither of the following

MOST x ALL y (if Loves xy then Quarrel xy)

(Suppose in a large domain there are few lovers.) nor

ALL y MOST x Quarrel xy
Love xy

(Suppose the domain is of three people: a loves a, b, c; b loves a, b, c; c loves no one. A pair quarrel iff they love. Remember MOST ALL is not equivalent to ALL MOST.)

There are also "cross scope" constructions in which a quantifier in the restriction of a quantifier binds a variable outside it. It is clear that the restriction of a quantifier can itself contain a quantifier, as in "All cats that chase most dogs are tough" or "Most cats that chase all dogs are ferocious". (The second in particular would not have the truth conditions it does if the restriction were to cats rather than to cats that chase all dogs.) There are sentences in which the quantifier syntactically within the restriction binds a variable occurring outside it. An example is "All cats that chase most dogs fear them". It is part of the specification of the individuals all of which fear most of the dogs they chase. I am taking this sentence so that the cats fear the very dogs they chase, not an equivalent proportion of them. One way of structuring the sentence preserving the scope relations, that MOST is part of the restriction of ALL is as

ALL x Fear xy
Cat x, MOST y *Chase xy*
Dog y

[FTN We cannot represent this sentence as

ALL x MOST y (MOST z Fear xz)
Cat x, Dog y, Dog z, Chase xy

Duplicating the MOST to avoid quantification past the restriction boundary, for that fails to specify that the chased cats are the feared cats.]

The analog of this case with two ALLs is interesting. Representing "All cats that chase all dogs fear them" as

ALL x Fear xy
Cat x, ALL y *Chase xy*
Dog y

and then applying the standard algorithm for absorbing the restriction into the scope clause, we get

∀x ((Cat.x &.∀y (Dog y.⊃.Chase.xy)) .⊃.Fear xy)

But this is not well-formed. We can fix this by putting the sentence in prenex form, as

∀x ∃y. ((Cat.x &. (Dog y.⊃.Chase.xy)) .⊃.Fear xy)

But this is not a mechanical application of the prenexing rules, which apply only to well-formed formulas. And there is room for doubt whether it captures the meaning of the English sentence. I leave this as a doubt about the standard line on restrictions of universal and existential quantifiers. It will return in another form in [the next] section.

We also can have quantifiers binding variables within the restrictions of other quantifiers ("most people who for a moment love everyone fear them at the same time"), and quantifiers in multiple restrictions binding variables within one another ("All cats that chase them fear most dogs that growl at most of these cats.") This is an interesting and puzzling phenomenon, and will be important in the [next] section. A simple case is "all cats care for most of their kittens". The obvious formalization is

ALL.x MOST.y Carefor.xy
Cat.x, Kitten.xy

But this presents a puzzle. For each cat x there is a set of x's kittens, and the sentence asserts that most of these kittens are cared for. This means that the quantifier MOST does not have a single restriction but a set of them, one for each cat. I take it to assert that for each cat x there is a set Y containing a majority of the kittens of x, and that x cares for every member y of this set. In a way, then the MOST is not a single quantifier with a single restriction, but a series of quantifiers with generally separate restrictions. An alternative idiom that brings out this side of the case is "All cats care for the majority of their kittens", which allows for different majorities for different cats. There are so many cases of this phenomenon – one is essential to the next section – that I do not think we can avoid accepting it as a feature of quantification in the wild.

Though I will not use the fact in what follows, it is worth noting that apparent violations of first order logic can result from a quantifier binding a variable in a restriction. For the extent of the restriction on of a quantifier Q' can depend on the value of a variable x bound by a quantifier Q in whose scope Q' lies. That is, we can have sentences

Qx Q'y Sxy
Ax, Rxy

An example: in the biography of a troubled person, t, we read "he had four major breakdowns. Each time everyone suffered horribly." From the narrative it is clear that different people suffered during his different breakdowns. His parents and siblings at first, his partners later, eventually his children. t himself suffered during two of the four, but the other two were euphoric and he was the one who did not. So Rxy here is "person y figures in t's life during breakdown x": who it includes depends on which breakdown we are considering. There is an apparent violation of the first order equivalence of "ALL.x ALL.y Rxy" and "All.y ALL.x Rxy" since it is not true that everyone suffered during each of t's breakdowns, if the wide-scope "everyone" is taken to mean everyone figuring in t's life.
[FTN: is it an apparent or a real violation? It would be real if we took first order logic to apply whatever the restriction, and included relational restrictions. For many this would be sufficient reason not to do so. But one could also work out a more subtle logic.

quantifier ordering

The focus of this paper is on ways of imposing ordering (scope) on quantifiers. This inevitably brings an emphasis on quantifiers in their "absolute" function, which specify how many individuals satisfy a relation's argument place rather than how many of those indexed by another quantifier satisfy it. For quantifiers not in the scope of others have to be absolute. In this section I describe a way of taking absolute quantifiers as basic and defining relative quantifiers in terms of them.

(a) *unordered* There are two basic types of multiple quantification. First there are cases where all the stated quantifiers are absolute and thus unordered, as in the examples from section [1]. In these it is clear that extra information is usually implicit or contextual. The information can often be given as relative quantifiers in the scope of the state absolute quantifiers, as in the example from section 1 where a standard meaning of "Six cats are chasing five dogs" has six cats chase five dogs, leaving it open how many dogs (all? one? a few? each cat chases) and how many cats chase each dog. The 'extra' quantifiers are best chosen so that negation is well-defined without a great change of vocabulary. There are two aspects to this. The extra quantifiers must be not very different from the duals of the explicit ones. (The dual of a quantifier 'Qx' is 'not Qx not', as with universal and existential quantifiers; the dual of a dual of a quantifier is the original quantifier; in appendix [///] I give a list of duals of familiar quantifiers.) And, the other aspect, for every pair of a quantifier Q and its dual <u>Q</u> we must construe one, normally the stronger one as a conjunction and the other as a disjunction. If these conditions are satisfied then "Most cats do not chase most dogs" is

[**] NOT (Most.x <u>Most</u>.z.Chase.xz & Most.y <u>Most</u>.w Chase.wy)
Cat x, Dog y

which is equivalent to

(<u>Most</u>.x Most.z NOT Chase.xz) OR <u>Most</u>.y Most.w NOT Chase.wy)

where <u>Most</u> is the dual of Most, in English "at least as many as not" ("a fair number", "considerably many", "quite a few", so from a little less than most to all.). The English for [the last of these] is "At least as many cats do not chase [most/very many] dogs as do", which is intuitively what the negation of [**] should be. I shall take this as the default form of a multiple quantification without an ordering of the stated quantifiers, in the absence of other information suggesting an alternative choice.

The choice of implicit relative quantifiers becomes less crucial when the quantifiers are vague or have a wide range. If we take "most" to apply definitely from 70 to 100% and less definitely from 51 to 69% then its dual applies definitely from 50 to 100% and less definitely from 31 to 49%. The dual of "at least a thousand" is "at

most 999". In both cases, and others like them, the formulation with the dual will entail formulations with a wide range of other quantifiers filling the implicit slots. So it is less important to pin down the exact quantifiers intended. Perhaps this is a reason why absolute constructions often use vague quantifiers ("many of the cats are chasing many of the dogs") while many constructions with explicit relative ones use precise ones ("each of exactly three cats chased exactly two dogs.")
[FTN the same would apply in terms of more sophisticated accounts of vagueness, allowing higher-order vagueness.]

In many cases explicit absolute quantifiers need to be supplemented by implicit relative ones. They do, that is, if we want to give the information supplied by the Frege-Tarski absolute-then-relative pattern. But that pattern leaves out information that absolute quantifiers can supply: "Each of two cats chased three dogs" leaves it open where between three and six the total number of chased dogs lies. You choose your idiom and then you improvise to communicate within its gaps. Elementary equivalence is a far cry from isomorphism.

(b) *linear ordering* The other basic type has an ordering for the stated quantifiers, where each relative quantifier is in the scope of an absolute quantifier. In the simplest case the quantifiers have a linear ordering, each in the scope of those to its 'left'. Consider first the case of ALL and SOME with a two-place relation R.

In the three cases of SOME.x ALL.y Rxy, SOME.x ALL.y Rxy, and All.x ALL.y Rxy, both quantifiers are absolute, since SOME occurs only in initial position and ALL is the same when absolute and when relative. That makes the prefixes in effect unordered, so that we could write them in the traditional form I have just used, or in a way that indicates lack of scope, such as

{SOME.x, ALL.y} Rxy   or $\begin{matrix}\text{SOME.x.}\\ \text{ALL.y}\end{matrix}$ Rxy

(See the remarks on this under *partial ordering* below.)

The remaining two SOME/ALL cases are SOME.x SOME.y Rxy and

ALL.x SOME.y Rxy. The first of these is unproblematic, since though only the first SOME is strictly speaking absolute, when there is an individual a such that Ra _ can be completed to a truth then any individual b making that completion, that is, such that Rab is true, will make SOME.y Rby true and hence SOME.y SOME.x Rxy true. [CLEARER?] So the sentence is true if and only if the two absolute SOME sentence, which we might write

{SOME.x, SOME.y} Rxy or SOME.x / SOME.y Rxy

is true.

[FTN: for simplicity I am saying that "Rab" is true rather than that <a,b> satisfies "Rxy", but the difference makes no difference here. The case could also be handled with a variable in a restriction, along the lines of the ALL.x SOME.y case below.]

That leaves ALL.x SOME.y Rxy to consider. Here the observation in the previous section of quantifiers binding variables in restrictions ("all cats care for a majority of their kittens") comes into play. Consider

ALL.x SOME.y Rxy
Ax, By, Rxy

This restricts values of y to individuals in B having R to x. So the effect is as if there is a series of absolute SOMEs, each indexed to a value of x. ("All As have R to their singularity of Bs". See the previous section.) (The analysis can also be explained as an absolute second order existential postulating a Skolem function **f** supplying a possibly different **fx** for each value of x.) In this way we can describe Frege-Tarski relative quantifiers in terms of absolute quantifiers.

[FTN: in the general case the restriction will involve the same R as in the scope clause. But in many special cases a simpler restriction will do. For example if the scope clause is "Sxy & Px" then the restriction can be to Sxy. There is a theorem waiting to be stated giving necessary and sufficient conditions given a complex matrix for a restriction that does the work of a Skolem function]

The same ideas apply to longer quantifier prefixes. For example to distinguish

ALL.x ALL.y SOME.z Rxyz

from

ALL.x SOME.z ALL.y Rxyz

we need to ensure that SOME.z is in the scope of both ALL quantifiers in [the first] but in the scope only of ALL.x in [the second.] The way to do this is to write [the first] with explicit restrictions as

{ALL.x, ALL.y, SOME.z} Rxyz
Ax, By, Cz, Rxyz

and [the second] as

{ALL., SOME.z, ALL.y} Rxyz
Ax, By, Cz, Some z Rxyz

[see theorem in appendix??]

(c) partial ordering Scope selectivity can be pushed further. In

ALL.x ALL.y SOME.z SOME.w Rxyzw
Ax, By, Cz, Dw, SOME.s SOME.t Rxszt, SOME.u SOME.v Ruyvw

z is independent of y and w is independent of z, so it is equivalent to the branching formula

ALL.x SOME.z
ALL.y SOME.w Fx & Gy & Hz & Iw & Rxyz

Ax, By, Cz, Dw

Any quantifier prefix that can be represented in linear form with Skolem functions can be represented in absolute-plus-restrictions form in this way. This should not be surprising, since the notation marks independence, leaving dependence, the target of Skolem functions, as the unmarked default.
(See appendix /// : the result is subject to one proviso, which I state there.)

Henkin invented the standard notation here, which I am adapting. Henkin only considered the universal and existential quantifiers, and only gave branching interpretations of quantifier blocks which themselves contain an existential quantifier within the scope of a universal. He could have considered branching pairs of universal or existential quantifiers, but he stuck with the standard notation which puts these in a row (at the price of arbitrary decisions about which is in the scope of which; see (b) above.) He could have considered the prefix which has ALL.x on one branch and SOME.y on another, but he did not since this is equivalent to the prefix

SOME.y ALL.x, and similarly for related cases, since his interest was in modifications that expand the power of first order logic. [FTN]

Henkin also did not consider branching quantifiers besides ALL and SOME. This is a topic has since attracted interest and which I discuss in the final section of this paper. Since this section discusses various orderings of quantifiers, though, I should mention that introducing a wider range of quantifiers makes some simple non-linear orderings possible that are equivalent to linear prefixes if we consider just ALL and SOME. For an example consider

ALL.x MOST.y / MOST.z MOST.w Rxyzw

which is not equivalent to
ALL.x MOST.y MOST.z MOST.w Rxyzw
[check!]

For many generalized quantifiers the same devices can be applied. For example "Most x have R to most y", with “Most y” in the scope of “Most x” is

{MOST.x, MOST.y} Rxy
Ax, By, Rxy

Branching quantifiers present a problem for absorbing restrictions of universal and existential quantifiers into scope clauses. The problem is that the technique for putting restrictions of ALL into the antecedents of a conditional puts some quantifiers found in restrictions within the scopes of others, and this may not be the intended sense. Here is an example.

When an Evans who loves all their brothers marries a Jones who loves all their sisters the brothers of the Evans and the sisters of the Jones are witnesses at their wedding

The form of this, with the restrictions as restrictions, is

ALL.x ALL.y Wxyzw
Ex, ALL.z (if Bxz then Lxz), Jy, ALL.w.(if Syw then Lyw)

Note that this has quantifiers in restrictions binding variables in the scope clause. Putting this in linear form in the usual way, we get something ill-formed, as with the example ("All cats that chase all dogs fear them") in the previous section. And as in that case the transition to prenex form is somewhat less problematic, yielding

[*] ∀x ∀y ∃z ∃w ((Ex & Jy & (Bxz.⊃.Lxz) & (Syw.⊃.Lyw)) .⊃. Wxyzw)

But now the two existential quantifiers are the scope of the two universals, so the choice of z is a function of both x and y and and that of w a function of x, while it should at least be an allowable meaning that each is governed only by the quantifier it restricts. (I would think this is the more natural interpretation: one spouse doesn't help choose the other spouse's witness. In fact I suspect such restrictions are one of the most promising sources for branching quantifiers in everyday speech.)

There are two ways of reacting to this. The first would be to make the quantifier prefix in [*] branch. That would mean that branching prefixes, and thus ALL/SOME sentences which are less first-order than they seem, are much less exotic than they are usually taken to be. The second would be to stick with the linear form and to accept that seemingly independent restrictions in fact are not. I think the first is the more plausible.

[FTN the example is meant to be simple but not to be one of the ALL ALL SOME SOME sentences for which the Skolem functions do separate into one-argument functions.]

donkey sentences

Geach's donkey sentence puzzle shows how idioms of natural language that are easily taken as standard Frege-Tarski universal and existential quantifiers may turn out to be something more general. Here are two sentences that raise the puzzle. Of a devout woman it is said

[//] If she has a son, she'll make him become a priest

Or, urging that a dangerous person be excluded from a demonstration, someone says

[\\] If he has a gun, he'll fire it at the police

The difficulty interpreting these sentences is that the first, for example, cannot have the form "∀x (Son.x .⊃. ∃y.Priest.xy)", as that would make it true if she has a son but she makes someone else become a priest ; it cannot have the form "∃x ∃y.(Son.x .⊃. Priest.xy)" as that makes it true given that there are things that are not her sons; and it cannot have the form "∀x ∀y (Son.x .⊃. Priest.xy)" as that would have the unintended consequence that if she has four sons she will make them all become priests. (And similarly for [\\] that if he has eight guns he'll fire all eight.) In addition to these hazards, there is a major desideratum, of making sense of the 'him' and the 'it' as legitimate anaphors rather than as linguistic quirks. (Parallel constructions are found in related languages: it is unlikely to be an unsystematic English idiom.)

[FTN not donkeys . confusing. and too much beating. Refs]

The solution, I believe, lies in thinking of the variety of things that quantifiers can range over, and how they can tune the sense of other quantifiers. Consider first some analogous sentences.

if the distance to the store is less than three kilometers , she'll walk it

if the number of feeding bowls is less than the number of cats, it won't be enough

In both of these 'it' is a genuine anaphoric pronoun, but one that refers to a parameter of a quantifier, the threshold it postulates for the number of individuals satisfying a criterion. Now consider an example nearer to our target class:

If she has one arrow remaining, she'll aim it at his heart

Again 'it' is a real anaphor, and tied to a quantifier meaning 'at least one', but this time it gives the appearance of referring to arrows rather than numbers of arrows. I suggest, though, that it does so in a way that is parasitic on the quantity parameter of the quantifier. That is, the sentence can be paraphrased as

If the number of her remaining arrows is one or greater then she'll take at least one and aim it at his heart

With this in mind, one line of solution is immediate. Suppose our devout mother will have only two sons. Then [//] is equivalent to

∀x ∀y ((∀z Son.z .⊃.(z=x v z=y)) .⊃. (Priest.x v Priest.y))

There are linked anaphors here. x and y range together over all her sons, and the number 2 is present in both antecedent and consequent. The general case is

[**]

$\forall n\ \forall x_1 \ldots \forall x_n\ (\forall z\ (Son.z\ .\supset.\ (z= x_1 \vee \ldots \vee z= x_n))\ .\supset.\ (Priest.\ x_1 \vee \ldots \vee Priest.\ x_n))$

See how the variables over her sons and the number variable combine. And note how the solution raises a question that has occurred twice already: when we have one quantifier and when we have a series of them. It would be nice to give a formulation without the indefinite series (the ...). Any such formulation will go beyond first order logic (as the quasi-infinitary [**] does.) One such would say that there is an ordering fom 0 to some n of all the sons, such that if a son comes first in the ordering then the mother makes him into a priest. Or more formally

[**o] $\exists n\ \exists o\ ([0\text{-}n]o\ \&\ \forall z.\exists m\ Romx\ \&\ \forall x\ (Ron.x\ .\supset\ Priest.x))$
Integer.n, Integer.m, Ordering.o

Here "[O-n]o" applies when o is an ordering of a set of individuals from 0 to n, and "Romx" applies when individual x has place m in ordering o.

There are versions of these that show how close to the surface form of [//] and [\\] they are.

[//*] If she has one or more sons, she will work on the first or a later one and make a priest of him

[*] If he has one or more guns, he'll take one and fire it at the police

'Him' in [//*] is now an anaphor referring both to to the number one and to a son that it indexes. 'It' in [*] refers to the number one and to the gun that the worrying person fires. The numerical-individual quantifier is prompted by the indefinite article "a"in [//] and [\\], which can serve both as "one" and as the individual quantifiers SOME and ALL, depending on context. (Otherwise similar sentences without "a", such as "if she has sons she will make them into priests", or "if he has exactly one gun, he'll fire it at the police", do not take the Geachian interpretation.) [//*] does not entail that only one son becomes a priest, or that all her sons do. [*] is not true if he has no guns or because there are non-guns. So they have the required features.

The requirement that the pronoun be "a real anaphor" excludes the interpretation [EE] "if she has a son then she will have a son she makes into a priest." ("If he has a gun then he will have a gun that he fires") in which a repeated existential quantifier does the work of "he" or "it". But a repeated quantifier is not a variable. (It is at most "one of them" rather than "it".) Still, the double existential sentence does tend to be true when the Geach sentence is. One way of bringing out the difference between them is to consider that [EE] "if she has a son then she will have a son she makes into a priest" is consistent with [EE~] "if she has a son then she will have a son she does not make into a priest." (She may have two sons.) On the other hand the combination of [//] "if she has a son she will make him into a priest" and [//~] "if she has a son she will not make him into a priest" is at the least peculiar and puzzling. A careful way to put this diference is that [//] and [//~] entail "she will have no son", while [EE] and [EE~] do not. This is another reason not to identify [//] and [EE].

This analysis puts these sentences, and donkey sentences in general, in a wide class of sentences with similar features. Examples are

If she has three sons then she'll make two into priests

If he has many guns then he'll fire several of them

If she has a dozen sons, she'll make most into priests

In all of these a smaller number of priests or firings is required by a larger number of sons or guns. It is worth noting that we use natural language quantifiers to refer to quantities as well as numbers, and that there too we use anaphoric pronouns in a way that at once refers to the quantity thresholds and to the substances of which these are quantities. For example

If he finds a pint or more of beer in the fridge he'll drink it in an hour

If one less gallon will bring the lake below the 1000 gallons needed to survive the summer then we had better not take it

If he has even a dollar more than a thousand he'll give *it* to you

The last of these is particularly interesting, as the "it" at first seems to refer to a particular dollar. But in fact it does not. (As a joke he might take a dollar off a pile and say "here's the one I've been saving for you.") This is like the reference to an integer or a threshold that disguises itself as a reference to a son or a gun.

All of these sentences can bear other non-Geachian interpretations, but so can the original donkey sentences. (Especially the original donkey sentences!) [///] can be read as requiring that all of the devout woman's sons are turned into priests. And as many have observed which reading a sentence with this syntax inclines to depends on details of context and vocabulary. (Though there are variants which are much less ambiguous, such as "if he has guns then he'll fire at least one of them at the police.")

The "Geach quantifier", Q_g [S,P] formalized as either [**] or as [**o] is not a first order quantifier. It cannot be represented as any combination of .∀,∃ and Boolean connectives. This is clear when one considers that "there are finitely many A" can be expressed in terms of its negation, with A substituted for S and
A & ~A substituted for P. ~ Q_g [A, A & ~A] is then

$$\exists n\ \exists x_1 \ldots \exists x_n\ (\forall z\ (Az\ .\supset.\ (z= x_1 \vee \ldots \vee z= x_n\ )) \ \&\ (Ax_1 \vee \sim Ax_1)\ \&\ \ldots\ \&\ (Ax_n \vee \sim Ax_n))$$

which is equivalent to

$$\exists n\ \exists x_1 \ldots \exists x_n\ (\forall z\ (Az\ .\supset.\ (z= x_1 \vee \ldots \vee z= x_n\ ))$$

As is well-known "there are finitely many A" cannot be expressed in first-order terms. It is interesting that both this and "there are at least as many A as B" can be expressed with branching quantifiers. [FTN: Henkin, Boolos, Krynicki & Lachlan]

conclusion

Hintikka raised the issue of whether branching quantifiers can give accurate formalizations of natural language sentences. In discussing Hintikka, and Fauconnier's criticism of Hintikka, Barwise introduced the idea of branching generalized quantifiers, particularly cases where we have a binary branching of two one-place quantifiers binding a two-place matrix, and Sher and others have refined and commented on his treatment. I have treated some such cases in terms of unordered quantifiers supplemented by implicit quantifiers in restrictions. To end the paper I shall contrast the branching quantifier view and the unordered quantifier accounts. Although we have a lot of freedom in what we can mean by quantified syntactical forms, and although we are quite ingenious in explaining which of the possibilities is the intended meaning, the unordered form is, I shall argue, the best fit for these cases.

A quantifier not in the scope of another quantifier normally takes an absolute interpretation, saying how many individuals from the domain in its restriction satisfy the relevant argument place of the scope clause, rather than the quantity for each individual in the range of a wider scope quantifier. So when we have a binary matrix (scope clause), we must have another quantifier, a relative one, specifying what objects have the relation to those invoked by the absolute one. Barwise's accommodation of this fact in his treatment of branching generalized quantifiers is to make the other quantifiers universal. He gives two definitions, both generalizing the use of Skolem functions in Henkin's approach:

Barwise 1) $\exists X \,.\, \exists Y \,[Q_1 x\ Xx \;\&\; Q_2 y\ Yy \;\&\; \forall x\, \forall y\, ((Xx \;\&\; Yy) \;.\!\supset\!.\; Sxy)]$
Ax, By

Barwise 2) $\exists X \,.\, \exists Y \,[Q_1 x\ Xx \;\&\; Q_2 y\ Yy \;\&\; \forall x\, \forall y\, (Sxy \;.\!\supset\!.\; (Xx \;\&\; Yy))]$
Ax, By

Barwise defends 1) for increasing quantifiers, where Qx Xx and $X \subseteq Y$ entail Qy Yy, and 2) for decreasing quantifiers, where Qx Xx and $Y \subseteq X$ entail Qy Yy. And indeed 1) makes little sense for decreasing Q nor 2) for increasing. So for MOST and ALL it is 1) that is relevant.

There are two problems with these definitions. First, they do not fit the meanings that sentences like "most of the cats are chasing most of the dogs" normally have, when neither "most" is taken as within the scope of the other. For "most", the Barwise version says that each of some majority set of cats chases each one of some majority set of dogs. This is too strong in two ways. The first is that it requires that each of these cats chase each of these dogs. But surely as we usually understand the sentence if the dogs chased by some members of a majority set of cats do not all fall into the [same?] majority set of dogs the sentence is not falsified. (We could take the union of the overlapping chased majorities, but this could create a set that was too large for all of its members to be chased by most chasers.) The second is that it requires that each cat chase a majority of the dogs. But surely if one cat chased one short of a majority the sentence would not be falsified, as long as most cats chase a fair number of dogs and most dogs are chased by a fair number of cats.

The other problem is that they are not closed under negation. The negations of (1) and (2) will not be of those forms. For one thing the existential set quantifiers will be turned into universals. So we are blocked from understanding e.g. "most of the cats are chasing most of the dogs" except as "it is not the case that most of the cats are chasing most of the dogs". We cannot understand it as "most of the cats are not chasing most of the dogs". This is a problem that affects Henkin's account too, though it is less severe as he does not aim at an analysis of natural language, and Hintikka's closely related "independence-friendly logic", where it is more severe since he does so aim. [FTN: when the branching is in the restrictions. see above]

Gila Sher has proposed an alternative to Barwise's analysis that is closer to the unordered treatment. On her account the non-linear application of Q_1 and Q_2 to a relation S is

$$Q_1x\ \exists z\ Sxz\ \&\ Q_2y\ \exists w\ Swy$$
Ax, By

So for example "Most cats fear most dogs", where neither MOST is in the scope of the other, would get an interpretation according to which it says that most cats fear some dog and most dogs are feared by some cat. Another way of putting it: considering all the pairs that satisfy S, the first members amount to most of A and the second members amount to most of B. Sher's interpretation has several advantages. It applies to both increasing and decreasing quantifiers, and indeed to quantifiers that are neither. And it is better behaved under negation, since it has no higher-order quantifiers. (There is a worry about the transformation of AND into

OR, though.)

Sher's definition has the disadvantage that it is very weak: often when we assert e.g. that most of the cats fear most of the dogs we mean more than this. A variant on Sher's definition, replacing the requirement that Q_1 A have S to some B and Q_2 B have S to some A with a requirement that Q_1 A have S to all B and Q_2 B have S to all A, would have the disadvantage of being too strong.

The analysis of two quantifier sentences defended in this paper, which I have been calling the unordered analysis, takes the full information non-linear application of Q_1 and Q_2 to a relation S to be

$$Q_1x\ \underline{Q_1z}\ Sxz\ \&\ Q_2y\ \underline{Q_2w}\ Swy$$

Ax ,By

where $\underline{Q}_i$ is the dual of Q_i. (I also argued that there are less than full information versions, which are perfectly intelligible.) So "most of the cats are chasing most of the dogs" becomes "most of the cats are chasing at least as many of the dogs as not, and most of the dogs are being chased by at least as many of the dogs as not." This is intermediate in strength between Sher's too-weak existential form and the alternative too-strong universal form. And in general the use of dual quantifiers will result in a formulation appropriately between these extremes. Moreover the analysis is closed under negation, especially when OR is taken as the dual of AND, as explained in section [|||].

Two or more unordered generalized quantifiers not in one another's scope can be treated as I have described. Two or more unordered universal and existential quantifiers not in one another's scope are equivalent to linear combinations of quantifiers, and thus do not branch in any interesting way. We could also have two or more strings of quantifiers, either generalized, universal or existential, so that the later quantifiers in each string are in the scope of earlier ones but the head absolute quantifiers in each string are not in one another's scope. Then we have genuine branching, as a distinct phenomenon from linear and unordered quantifiers. This could occur with generalized quantifiers. We could for example have

$$\begin{matrix}\text{Most.x} & \text{Some.y} \\ \text{All.z} & \text{Few.w}\end{matrix}\ \text{Rxyzw}$$

But these have not been proposed as candidates for the analysis of sentences in natural language. The unordered interpretation, with the option of additional implicit quantifiers as described, is a better fit for what we usually mean when we use expressions such as “most of the cats were chasing a few of the dogs”. The unordered interpretation is thus a promising candidate as an interpretation of quantified sentences where none of the explicitly stated quantifiers is within the scope of the others, and thus for an alternative to the Frege-Tarski analysis. Since the surface structure of natural language quantified sentences gives so few clues to their interpretation, though, it is worth repeating what a wide range of interpretations we are free to explain as our intentions. For one thing, mixtures of the possibilities I have been describing are possible. We can for example apply my favored unordered interpretation to combinations of Tarskian quantifiers, getting what we can write as

$$\{\forall x\ \exists y,\ \forall z\ \exists w\}\ Rxyzw$$
$$\textit{∃s∀t Rxyst}\quad \textit{∃s∀t Ruvzw}$$

and analyse as

$$\forall x \exists y \exists s \forall t\ Rxyst\ \&\ \forall z \exists w \exists s \forall t\ Ruvzw$$

Such a formulation might for some purpose be exactly what we need.

[note on Lindstrom's theorem: concerns taking a standard first order logic and adding more quantifiers. Does not say what happens if you withdraw and add or change basic features of first order syntax for example by separating restrictions from scope clauses. Further work.]

Appendix: duals

Q	=	~Q~	=
All		some	
Most	> 1/2	at least as many as not	1/2 ≤ n ≤all, 50% and up
Many	n > N	few not	
few (finite domain, proportional)	n < T/p	fairly many	n < sT/p
all but N (finite domain)	n > T+1-N		T+1-N
at least N (finite)	> N-1	within N of the maximum	>T-N T=max
exactly N	n = N	n < N or n > N+1	n not= N Not exactly N
infinitely many	all n |A|>n	cofinite A	some n |~A|<n
finitely many	some n |A|< n	"coinfinite"	all n |~A|>n
all but finitely many	some n |~A|< n	finitely many	some n |A|< n

appendix: counterexamples to equivalences with MOST, ALL, and related quantifiers

(a) "MOSTx (if Ax then Bx)" true but "MOSTx Bx" false:
Ax

D[omain] = {1, 2, 3 , 4, 5, 6, 7}, A = {1, 2, 3} B = {1}

(The entailment holds in the other direction.)

(b) " MOSTx ALLy Rxy" true and "ALLx MOSTy Rxy" false:
Ax, By Ax, By

D = {1, 2, 3}, A=B=D, R ={(1,1), (1,2), (1,3), (2,1), (2, 2), (2,3)}

(c) "ALLx MOSTy Rxy" true and "MOSTx ALLy Rxy" false:
Ax, By Ax, By

D = {1, 2, 3}, A=B=D, R ={(1,1), (1,2), (2,1), (2, 2), (3,1), (3,2)}

(in fact no A has R to all B.)

(d) "MOSTx MOSTy Rxy" true and "MOSTy MOSTx Rxy" false:
Ax, By Ax, By

D = {1, 2, 3}, A=B=D, R ={(1,1), (1,2), (2,1), (2,3)}

(e) "MOSTx MOSTy Rxy" and "ALLx MOSTy Rxy" true and "MOST(x,y) Rxy"
Ax, By Ax, By

false: D = {1, 2, a, b}, A = {1,2}, B = {a, b}, R ={(1,a), (1,b), (2,a), (2,b)}

MISLEADING. Should relativize counts to A, B. Then equivalence true for small n, but pattern changes for n greater 5 [non-equivalence true then]

(f) Let ALL^{-n} , for n > 0 ,mean "for all except n" (i.e. "for all with at most n exceptions"; ALL^{-0} is the standard universal quantifier.) Let ALL^{-fin} mean "for all except finitely many". In the model below all of (I) are true and all of (II) are false. (I suppress restrictions for simplicity.)

(I) ALL^{-fin} x ALL^{-fin} y Rxy

ALL^{-n} x ALL^{-fin} y Rxy

ALL x ALL^{-fin} y Rxy

(II) $ALL^{-fin\ y}$ ALL^{-fin} x Rxy

ALL^{-n}y ALL^{-fin} x Rxy

ALL^{-fin} y ALL x Rxy

ALL y ALL^{-fin} x Rxy

ALL y ALL^{-n} x Rxy

Model: D = {1,2,3, ...}, Rij iff i < j .

(For n=1, these can all be shown with 2-element models.)

(g) $\mathrm{MOST}_{Ax}\ \mathrm{MOST}_{By}$ (Sxy & **R**xy) & $\mathrm{MOST}_{Bz}\ \mathrm{MOST}_{Aw}$ (Swz & **R**wz)

true but MOST pairs (x,y) (Sxy & **R**x) false:

D = {1, 2, a, b}, A = {1,2}, B = {a, b}, R ={(1,a), (1,b), (2,a), (2,b)}

(Same model as in (e).)

biblio [notes, incomplete]

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